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1982 J. Phys. A: Math. Gen. 15 3453

(http://iopscience.iop.org/0305-4470/15/11/022)

J. Phys. A: Math. Gen. 15 (1982) 3453–3463. Printed in Great Britain

On the existence and uniqueness of non-commutative stochastic processes

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Received 8 October 1981, in final form 31 March 1982

Abstract. We introduce a general notion of non-commutative stochastic processes over complete locally convex *-algebras and prove a theorem concerning the existence and uniqueness of such processes. We show that quantum fields are stochastic processes in the sense of this paper. Moreover, we apply our theory to infer the existence and uniqueness of quantum fields.

1. Introduction

In this paper, we consider the notion of a non-commutative stochastic process in a rather general setting and we discuss the important question of the existence and uniqueness of such a process. A major aim of our presentation is to contribute to the foundation of a general theory of non-commutative stochastic processes. We see this as a worthwhile goal for, although non-commutative probabilistic methods have been employed by various authors in, for example, the discussion of non-commutative Markov fields (Ekhaguere 1979, Wilde 1974, Schrader and Uhlenbrock 1975), the existence and uniqueness of physical ground states (Gross 1972) and the quantum theory of open systems (Davies 1976), as far as this author is aware, unlike what obtains in the case of classical stochastic processes, there is as yet no general theory of non-commutative stochastic processes—a theory which is evidently desirable from both mathematical and physical points of view.

This work has been motivated by the recent presentation of Accardi *et al* (1981). Accordingly, the definition of a non-commutative stochastic process which we use in the sequel is analogous to the one introduced by Accardi (1976), but we do not require the algebras which appear in our formulation to be merely C^*- or W^*-algebras. The generalisation thus achieved is useful in, for example, quantum field theory (which is inherently probabilistic in a non-classical sense) where locally convex *-algebras, which are not necessarily C^*- or W^*-algebras, are encountered (Powers 1971, Lassner 1975).

The rest of the paper is organised as follows. Some of the concepts and notation which we employ in the sequel are introduced in § 2. In § 3, we develop our theory of non-commutative stochastic processes. This culminates in theorem 3.9 which concerns the existence and uniqueness of such processes. In § 4, we demonstrate that quantum fields are examples of stochastic processes in the sense of this paper, and we apply the theory to infer the existence and uniqueness of quantum fields.

0305-4470/82/113453+13$02.00

2. Basic concepts and notation

Throughout the paper, every algebra is assumed to be an algebra with identity. Furthermore, if $\mathfrak{A}$ is an algebra, we shall denote its underlying vector space by $\mathfrak{A}^{(0)}$.

In the sequel, we work with complete locally convex *-algebras. Recall that $\mathfrak{A}$ is such an algebra if its underlying vector space $\mathfrak{A}^{(0)}$ carries a locally convex topology with respect to which (i) the vector space $\mathfrak{A}^{(0)}$ is complete, (ii) the involution or *-operation of $\mathfrak{A}$ is continuous and (iii) the multiplication operation $(a_1a_2)\mapsto a_1a_2$ of $\mathfrak{A}\times\mathfrak{A}$ into $\mathfrak{A}$ is separately continuous in each of the two arguments.

Let $(\mathscr{A}, \leqslant)$ be a directed set (with $\leqslant$ as its ordering relation) and let $\{\mathscr{C}_\alpha : \alpha \in \mathscr{A}\}$ be a family of complete locally convex *-algebras. Then $\{\mathscr{C}_\alpha^{(0)} : \alpha\in\mathscr{A}\}$ has a locally convex direct sum which we denote by $\mathscr{C}^{(0)}$. The vector space $\mathscr{C}^{(0)}$ is the underlying linear space of a complete, locally convex *-algebra $\mathscr{C}$ which is the locally convex direct sum of the family $\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\}$.

Let g_α be the canonical *-homomorphic imbedding of $\mathscr{C}_\alpha$ in $\mathscr{C}$, $\alpha\in\mathscr{A}$. Then we shall denote by $\mathrm{Imb}(\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\})$ the collection of families

$$h = (h_{\beta\alpha} : (\alpha,\beta)\in\mathscr{A}\times\mathscr{A} \text{ and } \alpha\leqslant\beta)$$

of continuous linear mappings with the following properties: for each $h\in \mathrm{Imb}(\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\})$, with $h=(h_{\beta\alpha} : (\alpha,\beta)\in\mathscr{A}\times\mathscr{A}, \alpha\leqslant\beta)$, we have

(i) $h_{\beta\alpha}$ is a *-homomorphism of $\mathscr{C}_\alpha$ into $\mathscr{C}_\beta$, $\alpha\leqslant\beta$;

(ii) $h_{\alpha\alpha}$ is the identity *-automorphism of $\mathscr{C}_\alpha$;

(iii) $h_{\gamma\beta}\circ h_{\beta\alpha} = h_{\gamma\alpha}$, for $\alpha\leqslant\beta\leqslant\gamma$;

(iv) the locally convex direct sum *-algebra $\mathscr{C}_h$ generated by the family $\{(g_\alpha - g_\beta\circ h_{\beta\alpha})(\mathscr{C}_\alpha) : (\alpha,\beta)\in\mathscr{A}\times\mathscr{A}, \alpha\leqslant\beta\}$ is a *complete* *-subalgebra of $\mathscr{C}$.

It is evident that $\mathrm{Imb}(\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\})$ is never vacuous.

For $h\in\mathrm{Imb}(\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\})$, the quotient locally convex *-algebra $\mathscr{C}/\mathscr{C}_h$, which is obviously complete, is an inductive limit, determined by $h\in\mathrm{Imb}(\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\})$, of the family $\{\mathscr{C}_\alpha : \alpha\in\mathscr{A}\}$ of complete locally convex *-algebras. Inductive limits of complete locally convex *-algebras will be employed in the ensuing analysis.

3. Non-commutative stochastic processes

Let $\mathfrak{A}$ be a complete locally convex *-algebra and let $\mathfrak{A}^*$ be its topological dual. Then a normalised member μ of the positive cone $\mathfrak{A}^*_+$ of $\mathfrak{A}^*$ (i.e. a member $\mu\in\mathfrak{A}^*_+$ such that $\mu(\mathbb{1})=1$, where $\mathbb{1}$ is the identity of $\mathfrak{A}$) is called a state on $\mathfrak{A}$.

In the sequel, we shall refer to a pair $(\mathfrak{A},\mu)$, consisting of a complete locally convex *-algebra $\mathfrak{A}$ and a state μ on $\mathfrak{A}$, as a probability algebra.

3.1. *Definition.* Let $\{(\mathscr{B}_\alpha,\mu_\alpha) : \alpha\in\mathscr{A}\}$ be a family of probability algebras. We shall say that $\{\mu_\alpha : \alpha\in\mathscr{A}\}$ is a consistent family of states provided that there is some member $h=(h_{\beta\alpha} : (\alpha,\beta)\in\mathscr{A}\times\mathscr{A} \text{ and } \alpha\leqslant\beta)$ in $\mathrm{Imb}(\{\mathscr{B}_\alpha : \alpha\in\mathscr{A}\})$ such that $\mu_\alpha = \mu_\beta\circ h_{\beta\alpha}$ for all $(\alpha,\beta)\in\mathscr{A}\times\mathscr{A}$ with $\alpha\leqslant\beta$.

3.2. Remark. We shall now introduce the notion of stochastic process considered in this paper.

3.3 Definition. Let $(\mathfrak{A},\mu)$ and $\mathscr{B}$ be a probability algebra and a complete, locally convex *-algebra, respectively. By a (non-commutative) stochastic process over $(\mathfrak{A}, \mu)$ with values in $\mathscr{B}$ and indexed by an arbitrary set Γ, we refer to a family $\{\phi_\gamma: \gamma \in \Gamma\}$ of continuous *-homomorphisms of $\mathfrak{A}$ into $\mathscr{B}$.

3.4 Remark. Definition 3.3 generalises Accardi's notion (1976) of a non-commutative stochastic process since neither $\mathfrak{A}$ nor $\mathscr{B}$ is required to be merely a C^*-algebra or a W^*-algebra. This generalisation is helpful in the study of quantum fields. Recall that a quantum field is a one-parameter family (possessing one of the Schwartz test function spaces as its parameter or index set) of densely defined linear operators on some Hilbert space. See the examples in § 4.

3.5. The ordered set $\boldsymbol{\Gamma}$. Given the arbitrary set Γ, let Γ_n denote the n-fold Cartesian product of Γ with itself and set $\bigcup_{n=1}^{\infty} \Gamma_n = \boldsymbol{\Gamma}$. Then for each $\boldsymbol{\gamma}$ in $\boldsymbol{\Gamma}$, there is a positive integer $\#(\boldsymbol{\gamma})$ such that

$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{\#(\boldsymbol{\gamma})}).$$

The set $\boldsymbol{\Gamma}$ may be directed by means of a (reflexive, transitive and antisymmetric) relation $\leqslant$ defined on $\boldsymbol{\Gamma}\times\boldsymbol{\Gamma}$ as follows: for $(\boldsymbol{\gamma}, \boldsymbol{\gamma}') \in \boldsymbol{\Gamma}\times\boldsymbol{\Gamma}$, with $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{\#(\boldsymbol{\gamma})})$ and $\boldsymbol{\gamma}' = (\gamma_1', \gamma_2', \ldots, \gamma'_{\#(\boldsymbol{\gamma}')})$, we shall write $\boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}'$ if and and only if

(1) $\#(\boldsymbol{\gamma}) \leqslant \#(\boldsymbol{\gamma}')$, where $\leqslant$ is taken here as the natural ordering of the reals;

(2) $\{\gamma_1, \gamma_2, \ldots, \gamma_{\#(\boldsymbol{\gamma})}\} \subseteq \{\gamma_1', \gamma_2', \ldots, \gamma'_{\#(\boldsymbol{\gamma}')}\}$, and $\gamma_j' = \gamma_j$, $j = 1, 2, \ldots, \#(\boldsymbol{\gamma})$.

In the sequel, any reference (whether implicit or explicit) to an ordering of $\boldsymbol{\Gamma}$ shall be a reference to the relation $\leqslant$ just defined.

3.6. Probability distribution functionals. Let $\{\phi_\gamma: \gamma \in \Gamma\}$ be a $\mathscr{B}$-valued stochastic process on the probability algebra $(\mathfrak{A}, \mu)$. For $\boldsymbol{\gamma} \in \Gamma$, with $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{\#(\boldsymbol{\gamma})})$, let $\mathscr{B}_{\boldsymbol{\gamma}}$ denote the Cartesian product

$$\mathscr{B}_{\boldsymbol{\gamma}} = \phi_{\gamma_1}(\mathfrak{A}) \times \phi_{\gamma_2}(\mathfrak{A}) \times \ldots \times \phi_{\gamma_{\#(\boldsymbol{\gamma})}}(\mathfrak{A}).$$

We remark that $\mathscr{B}_{\boldsymbol{\gamma}}$ is a *-algebra in a natural way. Furthermore, $\mathscr{B}_{\boldsymbol{\gamma}}$ carries the induced product locally convex topology coming from the $\#(\boldsymbol{\gamma})$-fold cartesian product $\mathscr{B} \times \mathscr{B} \times \ldots \times \mathscr{B}$.

For $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, with $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{\#(\boldsymbol{\gamma})})$, let $\phi_{\boldsymbol{\gamma}}$ denote the *-homomorphism of $\mathfrak{A}$ onto $\mathscr{B}_{\boldsymbol{\gamma}}$ defined by

$$\mathfrak{A} \ni a \mapsto \phi_{\boldsymbol{\gamma}}(a) = (\phi_{\gamma_1}(a), \phi_{\gamma_2}(a), \ldots, \phi_{\gamma_{\#(\boldsymbol{\gamma})}}(a)) \in \mathscr{B}_{\boldsymbol{\gamma}}.$$

Denote the kernel of $\phi_{\boldsymbol{\gamma}}$ by $\ker(\phi_{\boldsymbol{\gamma}})$ and let $\pi_{\boldsymbol{\gamma}}$ be the canonical *-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\ker(\phi_{\boldsymbol{\gamma}})$. Then $\phi_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}$ is a *-isomorphism of $\mathfrak{A}$ onto $\mathscr{B}_{\boldsymbol{\gamma}}$. The map $\phi_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, induces a linear functional $\mu_{\boldsymbol{\gamma}}$ on $\mathscr{B}_{\boldsymbol{\gamma}}$ in the following way:

$$\mu_{\boldsymbol{\gamma}}(\boldsymbol{b}) = \mu((\phi_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}})^{-1}(\boldsymbol{b})), \qquad (\boldsymbol{b}, \boldsymbol{\gamma}) \in \mathscr{B}_{\boldsymbol{\gamma}} \times \boldsymbol{\Gamma}.$$

The net $\{\mu_{\boldsymbol{\gamma}}: \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ of linear functionals on the corresponding locally convex *-algebras $\{\mathscr{B}_{\boldsymbol{\gamma}}: \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ has the following obvious properties:

(1) $\mu_{\boldsymbol{\gamma}}(\mathbb{1}_{\boldsymbol{\gamma}}) = 1$, where $\mathbb{1}_{\boldsymbol{\gamma}}$ denotes the identity of $\mathscr{B}_{\boldsymbol{\gamma}}$;

(2) $\mu_{\boldsymbol{\gamma}}(\boldsymbol{b}^*\boldsymbol{b}) \geqslant 0$, for all $\boldsymbol{b}$ in $\mathscr{B}_{\boldsymbol{\gamma}}$;

(3) the mapping $(\boldsymbol{b}, \boldsymbol{c}) \mapsto \mu_{\boldsymbol{\gamma}}(\boldsymbol{b}^*\boldsymbol{c})$ of $\mathscr{B}_{\boldsymbol{\gamma}} \times \mathscr{B}_{\boldsymbol{\gamma}}$ into $\mathbb{C}$, the complex numbers, is a sesquilinear functional on $\mathscr{B}_{\boldsymbol{\gamma}} \times \mathscr{B}_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$.

In the sequel, the net $\{\mu_{\boldsymbol{\gamma}}: \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ of states will be referred to as the family of probability distribution functionals of the $\mathscr{B}$-valued stochastic process $\{\phi_\gamma: \gamma \in \Gamma\}$.

3.7. Equivalence of stochastic processes. Let $\mathfrak{A}$ be a complete locally convex *-algebra and let $\nu \in \mathfrak{A}^*_+$. Then there is (Powers 1971, 1974) a closed strongly cyclic *-representation ρ of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$, with inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$, and a strongly cyclic vector ξ in the domain of $\rho\ (\cdot)$ such that $\nu(a) = \langle\xi, \rho(a)\xi\rangle_{\mathscr{H}}$ for every $a \in \mathfrak{A}$.

We shall call $(\mathscr{H}, \xi, \rho)$ the Gelfand–Naimark–Segal (GNS) triple associated with the pair $(\mathfrak{A}, \nu)$, $\nu \in \mathfrak{A}^*_+$. The *-representation ρ is unique up to unitary equivalence.

Let $\{\phi_\gamma : \gamma \in \Gamma\}$ and $\{\phi'_\gamma : \gamma \in \Gamma\}$ be two stochastic processes over the same probability algebra $(\mathfrak{A}, \mu)$ but with values in the complete locally convex *-algebras $\mathscr{B}$ and $\mathscr{B}'$, respectively. Set $\pi_\gamma^{-1} \circ \phi_\gamma^{-1} \circ \phi_\gamma = \lambda_\gamma$ and $\pi_\gamma'^{-1} \circ \phi_\gamma'^{-1} \circ \phi'_\gamma = \lambda'_\gamma$, $\gamma \in \Gamma$.

Let $(\mathscr{H}_\gamma, \xi_\gamma, \rho_\gamma)$ and $(\mathscr{H}'_\gamma, \xi'_\gamma, \rho'_\gamma)$ be the GNS triples associated with $(\mathfrak{A}, \mu \circ \lambda_\gamma)$ and $(\mathfrak{A}, \mu \circ \lambda'_\gamma)$, respectively, for each $\gamma \in \Gamma$. Then we shall say that the $\mathscr{B}$-valued stochastic process $\{\phi_\gamma : \gamma \in \Gamma\}$ and the $\mathscr{B}'$-valued stochastic process $\{\phi'_\gamma : \gamma \in \Gamma\}$ over a common probability algebra $(\mathfrak{A}, \mu)$ are equivalent provided that there is a unitary transformation U_γ of $\mathscr{H}_\gamma$ onto $\mathscr{H}'_\gamma$ such that

$$\xi'_\gamma = U_\gamma \xi_\gamma \qquad \text{and} \qquad \rho'_\gamma(\cdot) = U_\gamma \rho_\gamma(\cdot) U_\gamma^{-1}$$

for each $\gamma \in \Gamma$.

3.8. Remark. We shall now state and prove our result in this paper.

3.9. Theorem. Let $\{\mathscr{B}_\gamma : \gamma \in \Gamma\}$ be a net of complete locally convex *-subalgebras of some fixed, complete, locally convex *-algebra. Let $\mathscr{B}$ be the complete locally convex *-algebra generated by $\bigcup_{\gamma \in \Gamma} \mathscr{B}_\gamma$. For $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{*(\gamma)})$ in $\boldsymbol{\Gamma}$, set

$$\mathscr{B}_{\gamma_1} \times \mathscr{B}_{\gamma_2} \times \ldots \times \mathscr{B}_{\gamma_{*(\gamma)}} = \mathscr{B}_{\boldsymbol{\gamma}}.$$

Let $\{(\mathscr{B}_{\boldsymbol{\gamma}}, \mu_{\boldsymbol{\gamma}}) : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ be a net of probability algebras indexed by the directed set $\boldsymbol{\Gamma}$. Suppose that the net $\{\mu_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ of states is a consistent family.

Then there exists a probability algebra $(\mathfrak{A}, \mu)$ and a $\mathscr{B}$-valued stochastic process $\{\phi_\gamma : \gamma \in \Gamma\}$ over $(\mathfrak{A}, \mu)$ which possesses $\{\mu_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ as its family of probability distribution functionals. Furthermore, the stochastic process $\{\phi_\gamma : \gamma \in \Gamma\}$ is unique up to equivalence.

Proof. Let $\mathscr{B}$ be the locally convex direct sum of the net $\{\mathscr{B}_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$. Then $\boldsymbol{\mathscr{B}}$ is, in an obvious way, a locally convex *-algebra which is automatically complete.

Since, by hypothesis, the net $\{\mu_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ of states is a consistent family, there is an $h = (h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}} : (\boldsymbol{\gamma}, \boldsymbol{\gamma}') \in \boldsymbol{\Gamma} \times \boldsymbol{\Gamma}, \boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}')$ in $\mathrm{Imb}(\{\mathscr{B}_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\})$ such that

$$\mu_{\boldsymbol{\gamma}} = \mu_{\boldsymbol{\gamma}'} \circ h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}}.$$

Let $g_{\boldsymbol{\gamma}}$ denote the canonical imbedding of $\mathscr{B}_{\boldsymbol{\gamma}}$ *in* $\boldsymbol{\mathscr{B}}$ and let $\boldsymbol{\mathscr{B}}_h$ be the locally convex *-algebra generated by the *-algebras

$$\{(g_{\boldsymbol{\gamma}} - g_{\boldsymbol{\gamma}'} \circ h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}})(\mathscr{B}_{\boldsymbol{\gamma}}) : (\boldsymbol{\gamma}, \boldsymbol{\gamma}') \in \boldsymbol{\Gamma} \times \boldsymbol{\Gamma}, \boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}'\}.$$

Since $h = (h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}} : (\boldsymbol{\gamma}, \boldsymbol{\gamma}') \in \boldsymbol{\Gamma} \times \boldsymbol{\Gamma}, \boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}')$ belongs to $\mathrm{Imb}(\{\mathscr{B}_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \Gamma\})$, by the definition of the latter, $\boldsymbol{\mathscr{B}}_h$ is automatically complete. Hence the quotient $\boldsymbol{\mathscr{B}}/\boldsymbol{\mathscr{B}}_h$ which is a complete locally convex *-algebra, is the inductive limit of the net $\{\mathscr{B}_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}$ determined by $h = (h_{\boldsymbol{\gamma}\boldsymbol{\gamma}} : (\boldsymbol{\gamma}, \boldsymbol{\gamma}') \in \boldsymbol{\Gamma} \times \boldsymbol{\Gamma}, \boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}')$ in $\mathrm{Imb}(\{\mathscr{B}_{\boldsymbol{\gamma}} : \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\})$. Put $\boldsymbol{\mathscr{B}}/\boldsymbol{\mathscr{B}}_h = \mathfrak{A}$. If b is a member of $\boldsymbol{\mathscr{B}}$, we shall denote by [b] the equivalence class of b in $\mathfrak{A}$.

For each $\boldsymbol{\gamma}$ in $\boldsymbol{\Gamma}$, the canonical *-homomorphic imbedding $g_{\boldsymbol{\gamma}}$ of $\mathscr{B}_{\boldsymbol{\gamma}}$ in $\boldsymbol{\mathscr{B}}$ induces a canonical *-homomorphic imbedding $\tilde{g}_{\boldsymbol{\gamma}}$ of $\mathscr{B}_{\boldsymbol{\gamma}}$ in $\mathfrak{A}$ defined thus:

$$\tilde{g}_{\boldsymbol{\gamma}}(\boldsymbol{b})=[g_{\boldsymbol{\gamma}}(\boldsymbol{b})],\ \boldsymbol{b}\in\mathscr{B}_{\boldsymbol{\gamma}}.$$

The maps $\{\tilde{g}_{\boldsymbol{\gamma}}\colon \boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ satisfy

$$\tilde{g}_{\boldsymbol{\gamma}}=\tilde{g}_{\boldsymbol{\gamma}'}\circ h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}}$$

for all $(\boldsymbol{\gamma},\boldsymbol{\gamma}')\in\boldsymbol{\Gamma}\times\boldsymbol{\Gamma}$ with $\boldsymbol{\gamma}\leqslant\boldsymbol{\gamma}'$. Hence in view of its consistency relative to $h=(h_{\boldsymbol{\gamma}'\boldsymbol{\gamma}}\colon(\boldsymbol{\gamma},\boldsymbol{\gamma}')\in\boldsymbol{\Gamma}\times\boldsymbol{\Gamma},\boldsymbol{\gamma}\leqslant\boldsymbol{\gamma}')$ in $\mathrm{Imb}(\{\mathscr{B}_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\})$, the net $\{\mu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ has an inductive limit which we denote by μ. Evidently, μ is a state on $\mathfrak{A}$. The pair $(\mathfrak{A},\mu)$ is the probability algebra whose existence was asserted.

Next denote the canonical *-homomorphism of $\boldsymbol{\mathscr{B}}$ onto $\mathfrak{A}=\boldsymbol{\mathscr{B}}/\boldsymbol{\mathscr{B}}_h$ by η. For $\boldsymbol{\gamma}\in\boldsymbol{\Gamma}$, let $P_{\boldsymbol{\gamma}}$ be the projection of $\boldsymbol{\mathscr{B}}$ onto $\boldsymbol{\mathscr{B}}_{\boldsymbol{\gamma}}$. Since $P_{\boldsymbol{\gamma}}$ is the projection of a locally convex direct sum onto one of its summand spaces, it is continuous. For $\gamma\in\Gamma$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_{*(\boldsymbol{\gamma})})\in\boldsymbol{\Gamma}$, with γ contained in the set $\{\gamma_1,\gamma_2,\ldots,\gamma_{*(\boldsymbol{\gamma})}\}$, let $P^{\boldsymbol{\gamma}}_{\gamma}$ be the projection of $\mathscr{B}_{\boldsymbol{\gamma}}$ onto $\mathscr{B}_{\gamma}$. Again, since $P^{\boldsymbol{\gamma}}_{\gamma}$ is the projection of a Cartesian product onto one of its factor spaces, it is continuous. The stochastic process $\{\phi_\gamma\colon\gamma\in\Gamma\}$ whose existence was asserted is now obtained by setting

$$\gamma_\gamma\equiv P^{\boldsymbol{\gamma}}_{\gamma}\circ P_{\boldsymbol{\gamma}}\circ\eta^{-1},\qquad \gamma\in\Gamma.$$

It is evident that ϕ_γ, $\gamma\in\Gamma$, is automatically continuous and is a *-homomorphism of $\mathfrak{A}$ into $\mathscr{B}$.

We claim that the family $\{\phi_\gamma\colon\gamma\in\Gamma\}$ possesses the net $\{\mu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ of states as its family of probability distribution functionals. To prove this, for $\boldsymbol{\gamma}\in\boldsymbol{\Gamma}$, with $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_{*(\boldsymbol{\gamma})})$, let $\phi_{\boldsymbol{\gamma}}$ be the mapping, of $\mathfrak{A}$ into $\mathscr{B}_{\boldsymbol{\gamma}}=\mathscr{B}_{\gamma_1}\times\mathscr{B}_{\gamma_2}\times\ldots\times\mathscr{B}_{\gamma_{*(\boldsymbol{\gamma})}}$, given by

$$a\mapsto\phi_{\boldsymbol{\gamma}}(a)=(\phi_{\gamma_1}(a),\phi_{\gamma_2}(a),\ldots,\phi_{\gamma_{*(\boldsymbol{\gamma})}}(a)),\qquad a\in\mathfrak{A},$$

and denote the range of $\phi_{\boldsymbol{\gamma}}$ by $\mathrm{ran}(\phi_{\boldsymbol{\gamma}})$. Let $\pi_{\boldsymbol{\gamma}}$ denote the canonical *-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\ker(\phi_{\boldsymbol{\gamma}})$. Let $\{\nu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ be the net of probability distribution functionals of $\{\phi_\gamma\colon\gamma\in\Gamma)$. Then by (3.6)

$$\nu_{\boldsymbol{\gamma}}(b)=\mu((\phi_{\boldsymbol{\gamma}}\circ\pi_{\boldsymbol{\gamma}})^{-1}(\boldsymbol{b})),\qquad \boldsymbol{b}\in\mathrm{ran}(\phi_{\boldsymbol{\gamma}}).$$

But $(\phi_{\boldsymbol{\gamma}}\circ\pi_{\boldsymbol{\gamma}})^{-1}$ is merely the restriction to $\mathrm{ran}(\phi_{\boldsymbol{\gamma}})$ of the canonical imbedding $\tilde{g}_{\boldsymbol{\gamma}}$ of $\mathscr{B}_{\boldsymbol{\gamma}}$ into $\mathfrak{A}$. Hence

$$\nu_{\boldsymbol{\gamma}}(\boldsymbol{b})=(\mu\circ\tilde{g}_{\boldsymbol{\gamma}})(\boldsymbol{b}),\qquad \text{for all } \boldsymbol{b}\in\ \mathrm{ran}(\phi_{\boldsymbol{\gamma}}).$$

But, since μ is the inductive limit of the net $\{\mu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ of states, it follows that

$$\mu\circ\tilde{g}_{\boldsymbol{\gamma}}=\mu_{\boldsymbol{\gamma}}.$$

Hence $\{\mu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ is indeed the family of probability distribution functionals of $\{\phi_\gamma\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$. The family $\{\phi_\gamma\colon\gamma\in\Gamma\}$ is a $\mathscr{B}$-valued stochastic process whose existence was asserted.

Finally, to prove uniqueness up to equivalence, let $\mathscr{B}'$ be some other complete locally convex *-algebra and suppose that $\{\phi'_\gamma\colon\gamma\in\Gamma\}$ is a $\mathscr{B}'$-valued stochastic process over $(\mathfrak{A},\mu)$ which also possesses the net $\{\mu_{\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}$ as its family of probability distribution functionals.

Now, if $\lambda_\gamma = \pi_\gamma^{-1} \circ \phi_\gamma^{-1} \circ \phi_\gamma$ and $\lambda'_\gamma = \pi'^{-1}_\gamma \circ \phi'^{-1}_\gamma \circ \phi'_\gamma$, $\gamma \in \Gamma$, then the nets $\{\mu \circ \lambda_\gamma: \gamma \in \Gamma\}$ and $\{\mu \circ \lambda'_\gamma: \gamma \in \Gamma\}$ of states on $\mathfrak{A}$ are the probability distribution functionals of the $\mathscr{B}$-valued stochastic process $\{\phi_\gamma: \gamma \in \Gamma\}$ and the $\mathscr{B}'$- valued stochastic process $\{\phi'_\gamma: \gamma \in \Gamma\}$, respectively, over $(\mathfrak{A}, \mu)$.

Let $(\mathscr{H}_\gamma, \xi_\gamma, \rho_\gamma)$ and $(\mathscr{H}'_\gamma, \xi'_\gamma, \rho'_\gamma)$ be the GNS triples associated with the probability algebras $(\mathfrak{A}, \mu \circ \lambda_\gamma)$ and $(\mathfrak{A}, \mu \circ \lambda'_\gamma)$, respectively, $\gamma \in \Gamma$. Define the linear map

$$U_\gamma: \mathscr{H}_\gamma \to \mathscr{H}'_\gamma$$

as follows:

$$U_\gamma \rho_\gamma(a_1)\rho_\gamma(a_2) \dots \rho_\gamma(a_n)\xi_\gamma = \rho'_\gamma(a_1)\rho'_\gamma(a_2) \dots \rho'_\gamma(a_n)\xi'_\gamma$$

for arbitrary $(a_1, a_2, \dots, a_n)$ belonging to the n-fold Cartesian product of $\mathfrak{A}$ with itself. Since the vectors ξ_γ and ξ'_γ, $\gamma \in \Gamma$, are strongly cyclic, U_γ extends to a continuous linear transformation of $\mathscr{H}_\gamma$ onto $\mathscr{H}'_\gamma$. Using the fact that

$$(\mu \circ \lambda'_\gamma)(a) = (\mu \circ \lambda_\gamma)(a), \qquad \text{for all } a \in \mathfrak{A},$$

it follows that U_γ is unitary, and we have

$$\xi'_\gamma = U_\gamma \xi_\gamma$$

and

$$\rho_\gamma(\cdot) = U_\gamma \rho_\gamma(\cdot) U_\gamma^{-1}, \qquad \gamma \in \Gamma.$$

This concludes the proof of the equivalence of the stochastic processes $\{\phi_\gamma: \gamma \in \Gamma\}$ and $\{\phi'_\gamma: \gamma \in \Gamma\}$ over $(\mathfrak{A}, \mu)$ with values in $\mathscr{B}$ and $\mathscr{B}'$, respectively. □

3.10. Remark. The general theory given above may be compared with the treatment presented by Accardi *et al* (1981) who introduce the notion of correlation kernels.

4. Some examples of stochastic processes

4.1. Quantum fields as examples of stochastic processes

Let $\mathbb{R}$ be the real line and $\mathbb{R}^d$ be the d-fold Cartesian product of $\mathbb{R}$ with itself. We shall denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space (Schwartz 1957/9) of complex-valued C^∞ functions which are rapidly decreasing at infinity and we shall write $\mathscr{S}(\mathbb{R}^d)'$ for the topological dual of $\mathscr{S}(\mathbb{R}^d)$.

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We shall say that φ is an operator-valued distribution with domain in $\mathscr{H}$ provided that:

(i) for each f in $\mathscr{S}(\mathbb{R}^d)$, $\varphi(f)$ is a densely defined linear operator with dense domain $\mathscr{D}(\varphi(f))$ in $\mathscr{H}$;

(ii) the mapping $f \mapsto \langle \varphi(f)\xi, \eta \rangle$ of $\mathscr{S}(\mathbb{R}^d)$ into $\mathbb{C}$ is in $\mathscr{S}(\mathbb{R}^d)'$ for arbitrary ξ, η belonging to $\mathscr{D}(\varphi(f))$;

(iii) the domain $\mathscr{D}(\varphi(f))$ is invariant for $\varphi(f)$;

(iv) the family $\varphi(\mathscr{S}) = \{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\}$ has a common dense invariant domain in $\mathscr{H}$ which we denote by $\mathscr{D}$.

If $\varphi(f)$ and $\varphi(g)$, $f, g \in \mathscr{S}(\mathbb{R}^d)$, are arbitrary members of $\varphi(\mathscr{S})$, then their sum $\varphi(f) + \varphi(g)$ and their product $\varphi(f)\varphi(g)$ will be taken in the sequel to be defined by

$$(\varphi(f) + \varphi(g))\xi = \varphi(f)\xi + \phi(g)\xi, \quad \varphi(f)\varphi(g)\xi = \varphi(f)(\varphi(g)\xi), \qquad \xi \in \mathscr{D}.$$

In quantum field theory, quantum fields are operator-valued distributions which satisfy certain assumptions called the Wightman axioms (Jost 1965, Streater and Wightman 1964). For simplicity, in the sequel, we shall only demonstrate that scalar quantum fields are examples of stochastic processes, as formulated in this paper.

Let $\mathscr{B}_0$ be the polynomial algebra generated by $\varphi(\mathscr{S})$, i.e. an arbitrary member p of $\mathscr{B}_0$ is of the form

$$p=\alpha_0\mathbb{1}+\sum_{n=1}^{N}\alpha_n\varphi(f_1)\varphi(f_2)\ldots\varphi(f_n)$$

where $\mathbb{1}$ is the identity operator on $\mathscr{H}$, $f_i\in\mathscr{S}(\mathbb{R}^d)$, $i=1,2,\ldots,n$ and $\alpha_j\in\mathbb{C}$, $j=0,1,2,\ldots,N$. We shall denote by $\mathscr{B}$ the completion of $\mathscr{B}_0$ in the locally convex topology generated by the family $\{|||\cdot|||_D\colon D$ is a finite subject of $\mathscr{D}\}$ of seminorms defined by

$$|||p|||_D=\sup\{|\langle p\xi,\eta\rangle|\colon(\xi,\eta)\in D\times D\},\qquad p\in\mathscr{B}_0.$$

In this topology, the operation * of taking the operator adjoint of a member of $\mathscr{B}$ is continuous. Hence $\mathscr{B}$ is a complete locally convex *-algebra with $\mathbb{1}$ as its identity element. We shall work with the algebra $\mathscr{B}$ in the sequel.

Next, let φ be an operator-valued distribution with domain in $\mathscr{H}$; symbolically, $\varphi(f)$ may be written as $\varphi(f)=\int \mathrm{d}x\,\varphi(x)f(x)$, $f\in\mathscr{S}(\mathbb{R}^d)$. For $n\geqslant 1$, let φ_n denote the n-fold tensor product of φ with itself. Define φ_0 as the operator-valued distribution which may be written symbolically as $\varphi_0(f)=(\int \mathrm{d}x\,(f(x))\mathbb{1}$, $f\in\mathscr{S}(\mathbb{R}^d)$, where $\mathbb{1}$ is the identity operator on $\mathscr{H}$. Notice that if f_n belongs to the n-fold tensor product $\mathscr{S}_n(\mathbb{R}^d)$ of $\mathscr{S}(\mathbb{R}^d)$ with itself, and $f_n=\otimes_{i=1}^n f_{ni}$, then $\varphi_n(f_n)=\varphi(f_{n1})\varphi(f_{n2})\ldots\varphi(f_{nn})$ which is a member of $\mathscr{B}$.

In what follows, we write $\mathfrak{A}_{00}$ for the collection of all elements of the form $\oplus_{k=0}^n\alpha_k\varphi_k$, where $\alpha_k\in\mathbb{C}$, $k=0,1,2,\ldots$, and $\oplus$ denotes direct sum. With $\oplus$ as addition and $\otimes$ as multiplication, $\mathfrak{A}_{00}$ is evidently an algebra which may be called the polynomial tensor algebra generated by φ. Furthermore, $\mathfrak{A}_{00}$ is a *-algebra with involution * defined as follows. For arbitrary $\oplus_{k=0}^n\alpha_k\varphi_k$ belonging to $\mathfrak{A}_{00}$, define $(\oplus_{k=0}^n\alpha_k\varphi_k)^*$ by

$$\left(\bigoplus_{k=0}^n\alpha_k\varphi_k\right)^*\left(\bigoplus_{k=0}^n f_k\right)=\sum_{k=0}^n\bar{\alpha}_k\varphi_k(\bar{f}_k)$$

where $f_0\in\mathscr{S}(\mathbb{R}^d)$, $f_k\in\mathscr{S}_k(\mathbb{R}^d)$, $\bar{f}_k$ is the complex conjugate of f_k, $k=1,2,\ldots,n$, and $\alpha_j\in\mathbb{C}$, $j=0,1,2,\ldots,n$.

We remark that the *-algebra $\mathfrak{A}_{00}$ is not endowed with an identity element.

Let $\mathfrak{A}_0$ be the completion of $\mathfrak{A}_{00}$ in the locally convex topology determined by the family $\{\|\cdot\|_D\colon D$ is a finite subset of $\mathscr{D}\}$ of seminorms defined by

$$\left\|\bigoplus_{k=0}^n\alpha_k\varphi_k\right\|_D=\sup\left\{\left|\left\langle\left(\bigoplus_{k=0}^n\alpha_k\varphi_k\right)\left(\bigoplus_{k=0}^n f_k\right)\xi,\eta\right\rangle\right|\colon(\xi,\eta)\in D\times D\text{ and }\bigoplus_{k=0}^n f_k\in\bigoplus_{k=0}^n\mathscr{S}_k(\mathbb{R}^d),\text{ with }\mathscr{S}_0(\mathbb{R}^d)=\mathscr{S}_1(\mathbb{R}^d)=\mathscr{S}(\mathbb{R}^d)\right\}.$$

Then $\mathfrak{A}_0$ is a complete locally convex *-algebra without identity.

Let $\mathfrak{A}$ be the unitification of $\mathfrak{A}_0$, i.e. $\mathfrak{A}$ is the algebra obtained from $\mathfrak{A}_0$ by adjoining an identity element to $\mathfrak{A}_0$. Recall that $\mathfrak{A}$ is merely the Cartesian product $\mathbb{C}\times\mathfrak{A}_0$ equipped with the product topology and that the element $\mathbb{1}=(1,0)\in\mathbb{C}\times\mathfrak{A}_0$ is the identity of $\mathfrak{A}$. It is noteworthy that $\mathfrak{A}$ is a complete locally convex *-algebra (with identity $\mathbb{1}$) in a natural way.

Let μ be a state on $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is a probability algebra.

We now define a stochastic process $\{\phi_f: f \in \mathscr{S}(\mathbb{R}^d)\}$ over $(\mathfrak{A}, \mu)$ with values in $\mathscr{B}$ and indexed by $\mathscr{S}(\mathbb{R}^d)$ as follows: for arbitrary $(\beta, \oplus_{k=0}^n \alpha_k \varphi_k)$ in $\mathbb{C} \times \mathfrak{A}_{00}$, set $\phi_f((\beta, \oplus_{k=0}^n \alpha_k\varphi_k)) = \beta\mathbb{1} + \alpha_0\varphi_0(f) + \Sigma_{k=1}^n \alpha_k(\varphi(f))^k$; in particular, $\phi_f((0, \varphi)) = \varphi(f), f \in \mathscr{S}(\mathbb{R}^d)$. Since $\mathbb{C} \times \mathfrak{A}_{00}$ is dense in $\mathfrak{A} = \mathbb{C} \times \mathfrak{A}_0$, ϕ_f has a unique extension (denoted again by ϕ_f) to all of $\mathfrak{A}$. Notice that ϕ_f takes the identity element $1 = (1, 0)$ of $\mathfrak{A}$ to the identity element $\mathbb{1}$ of $\mathscr{B}$. Moreover, it is evident that ϕ_f is a continuous *-homomorphism of $\mathfrak{A}$ into $\mathscr{B}$ for each $f \in \mathscr{S}(\mathbb{R}^d)$.

Assume that the operators $\varphi(\mathscr{S}) = \{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\}$ satisfy the Wightman axioms, so that $\varphi(\mathscr{S})$ is a scalar quantum field. Then one now sees how the latter arises in our formulation: the operators $\varphi(\mathscr{S}) = \{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\}$ lie in the range (which is contained in $\mathscr{B}$) of the two-sided *-ideal $\mathfrak{A}_0$ of $\mathfrak{A}$ under the action of the family $\{\phi_f: f \in \mathscr{S}(\mathbb{R}^d)\}$ of *-homomorphisms.

4.2. Existence and uniqueness of quantum fields

We shall next demonstrate how the existence and uniqueness of quantum fields may be inferred from our theory. Again, for the sake of simplicity, we confine the discussion to the case of scalar quantum fields.

Set $\bigcup_{n=1}^{\infty} \mathscr{S}_n(\mathbb{R}^d) = \boldsymbol{\mathscr{S}}$. In the sequel, we shall suppose that $\boldsymbol{\mathscr{S}}$ has been endowed with an ordering relation $\leqslant$ as described in (3.5). Then $(\boldsymbol{\mathscr{S}}, \leqslant)$ is a directed set.

Let $\mathscr{B}$ be an arbitrary complete locally convex *-algebra with an identity element $\mathbf{1}$. For each $f \in \mathscr{S}(\mathbb{R}^d)$, set $\mathscr{B}_f = \mathscr{B}$ and if f belongs to $\boldsymbol{\mathscr{S}}$, with $f = (f_1, f_2, \ldots, f_{*(f)})$, define $\mathscr{B}_f$ by $\mathscr{B}_f = \mathscr{B}_{f_1} \times \mathscr{B}_{f_2} \times \ldots \times \mathscr{B}_{f_{*(f)}}$. It is evident that $\mathscr{B}_f$ is, in an obvious way, a complete locally convex *-algebra with an identity element, which we denote by $\mathbb{1}_f$. Multiplication of two members of $\mathscr{B}_f$ will be denoted by juxtaposition.

Let $\boldsymbol{\mathscr{B}} = \bigcup_{f \in \boldsymbol{\mathscr{S}}} \mathscr{B}_f$ and $M(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}}) = \{$all mappings from $\boldsymbol{\mathscr{S}}$ into $\boldsymbol{\mathscr{B}}\}$. For each $f \in \boldsymbol{\mathscr{S}}$, define $M_f(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}})$ by

$$M_f(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}}) = \{b \in M(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}}): b(f) \in \mathscr{B}_f\}.$$

We remark that $M_f(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}})$ is a *-algebra if we define $\alpha b^{(1)} + \beta b^{(2)}$, $(\alpha, \beta)| \in \mathbb{C}^2$, $b^{(1)}b^{(2)}$ and b^* as follows:

$$(\alpha b^{(1)} + \beta b^{(2)})(f) = \alpha b^{(1)}(f) + \beta b^{(2)}(f),$$

$$(b^{(1)}b^{(2)})(f) = b^{(1)}(f)b^{(2)}(f), \qquad (b^*)(f) = b(f)^*,$$

for arbitrary $b, b^{(1)}, b^{(2)} \in M_f(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}})$.

In what follows, $\mathbb{1}$ will denote the member of $M_f(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}})$ defined by $\mathbb{1}(f) = \mathbb{1}_f$.

Since $\mathscr{B}_f$ is a *-algebra, it follows that $\mathscr{B}_f^* = \mathscr{B}_f$. Notice too that $\mathscr{B}_{\bar{f}} = \mathscr{B}_f$, where $\bar{f} = (\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{*(f)})$ is the complex conjugate of $f \in \boldsymbol{\mathscr{S}}$. Thus $\mathscr{B}_f^* = \mathscr{B}_{\bar{f}}$. Now in what follows, we make the assumption that

$$b(f)^* = b(\bar{f}) \tag{4.1}$$

for each $b(f) \in \mathscr{B}_f$ with $f \in \boldsymbol{\mathscr{S}}$.

In the subsequent discussion, we shall employ a family $W(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}}) \equiv \{W_b^{(\nu)}: \nu \in \mathbb{N}_+ = $ the positive natural numbers, $b \in M(\boldsymbol{\mathscr{S}}, \boldsymbol{\mathscr{B}}), f \in \boldsymbol{\mathscr{S}}\}$ of (Schwartz) distributions which possesses the following properties:

(i) for each $\nu \in \mathbb{N}_+$ and $b \in M_f(\mathscr{S}, \mathscr{B})$, $b \neq \mathbb{1}$, $W_b^{(\nu)}$ is a distribution on the ν-fold Cartesian product of $\mathscr{S}_{*(f)}(\mathbb{R}^d)$ with itself;

(ii) for each $\nu \in \mathbb{N}_+$, $W_b^{(\nu)}$ depends linearly on $b \in M_f(\mathscr{S}, \mathscr{B})$, $b \neq \mathbb{1}$, $f \in \mathscr{S}$;

(iii) for each $b \in M_f(\mathscr{S}, \mathscr{B})$, $f \in \mathscr{S}$, the distribution $W_{b^*b}^{(2)}$, $b \neq \mathbb{1}$ is non-negative in the sense that

$$W_{b^*b}^{(2)}(\bar{f}, f) \geq 0,$$

for every $f \in \mathscr{S}$, $b \neq \mathbb{1}$;

(iv) $W_{\mathbb{1}}^{(\nu)}(f, f, \ldots, f) = 1$, for all $(f, f, \ldots, f) \in (\mathscr{S}_{*(f)}(\mathbb{R}^d))^{\nu}$;

(v) if $f^{(1)}, f^{(2)}$ belong to $\mathscr{S}$ with $f^{(1)} \leq f^{(2)}$, $a \in M_{f^{(1)}}(\mathscr{S}, \mathscr{B})$ with $a(f^{(1)}) = (a_{f_1^{(1)}}, a_{f_2^{(1)}}, \ldots, a_{f_{*(f^{(1)})}})$ and $b \in M_{f^{(2)}}(\mathscr{S}, \mathscr{B})$ with $b(f^{(2)}) = (a_{f_1^{(1)}}, a_{f_2^{(1)}}, \ldots, a_{f_{*(f^{(1)})}}, \mathbf{1}, \ldots, \mathbf{1})$, then $W_b^{(1)}(f^{(2)}) = W_a^{(1)}(f^{(1)})$.

4.2. *Definition.* For each $f \in \mathscr{S}$, define a linear functional μ_f on $\mathscr{B}_f$ by requiring that for any $\nu \geq 1$ arbitrary members $b^{(1)}(f), b^{(2)}(f), \ldots, b^{(\nu)}(f)$ belonging to $\mathscr{B}_f$,

$$\mu_f(b^{(1)}(f)b^{(2)}(f) \ldots b^{(\nu)}(f)) = W_{b^{(1)}b^{(2)}\ldots b^{(\nu)}}^{(\nu)}(f, f, \ldots, f), \qquad f \in \mathscr{S}.$$

4.3. *Remark*

(i) Notice that as a result of the properties of members of $W(\mathscr{S}, \mathscr{B})$, μ_f is a state on $\mathscr{B}_f$, i.e. the pair $(\mathscr{B}_f, \mu_f)$ is a probability algebra for each $f \in \mathscr{S}$.

(ii) For $f^{(1)}, f^{(2)} \in \mathscr{S}$, with $f^{(1)} \leq f^{(2)}$, define a *-homomorphism $h_{f^{(2)}f^{(1)}}$ of $\mathscr{B}_{f^{(1)}}$ into $\mathscr{B}_{f^{(2)}}$ as follows: for

$$a(f^{(1)}) \in \mathscr{B}_{f^{(1)}},$$

with

$$a(f^{(1)}) = (a_{f_1^{(1)}}, a_{f_2^{(1)}}, \ldots, a_{f^{(1)}_{*(f^{(1)})}}),$$

set

$$h_{f^{(2)}f^{(1)}}(a(f^{(1)})) = (a_{f_1^{(1)}}, a_{f_2^{(1)}}, \ldots, a_{f_{*(f^{(1)})}}, \mathbf{1}, \ldots, \mathbf{1}) \in \mathscr{B}_f^{(2)}.$$

Then as a consequence of property (v) of the collection $W(\mathscr{S}, \mathscr{B})$ of distributions, we see that the net $\{\mu_f : f \in \mathscr{S}\}$ of states is consistent relative to the family of *-homomorphisms $\{h_{f^{(2)}f^{(1)}} : (f^{(1)}, f^{(2)}) \in \mathscr{S} \times \mathscr{S} \text{ with } f^{(1)} \leq f^{(2)}\}$.

(iii) We may now state and prove our final result concerning the existence and uniqueness of scalar quantum fields.

4.4. *Theorem.* Let $\{(\mathscr{B}_f, \mu_f) : f \in \mathscr{S}\}$ be the net of probability algebras defined as in the foregoing. Then there exists

(i) a Hilbert space $(\mathscr{H}, \langle\, , \rangle_{\mathscr{H}})$

(ii) a family $\{\varphi(f) : f \in \mathscr{S}(\mathbb{R}^d)\}$ of linear operators which, together with their adjoints, possess a common, dense, invariant domain $\mathscr{D}$ in $\mathscr{H}$, and

(iii) a unique vector $\xi_0 \in \mathscr{D}$, such that

$$\langle \xi_0, \varphi(f_1)\varphi(f_2) \ldots \varphi(f_{*(f)})\xi_0 \rangle_{\mathscr{H}} = \mu_f(b_0(f))$$

for some fixed member $b_0(f) \in \mathscr{B}_f$ and for all $f \in \mathscr{S}$.

Moreover, in the case where ξ_0 is cyclic for the polynomial *-algebra $P(\varphi)$ generated by $\{\varphi(f) : f \in \mathscr{S}(\mathbb{R}^d)\}$, the quadruplet $((\mathscr{H}, \langle\, , \rangle_{\mathscr{H}}), \mathscr{D}, \xi_0, \{\varphi(f) : f \in \mathscr{S}(\mathbb{R}^d)\})$ is unique up to unitary equivalence. That is, if $((\mathscr{H}', \langle\, , \rangle_{\mathscr{H}'}), \mathscr{D}', \xi_0', \{\varphi'(f) : f \in \mathscr{S}(\mathbb{R}^d)\})$ is another quadruplet determined by some net $\{(\mathscr{B}_f', \mu_f') : f \in \mathscr{S}\}$ of probability algebras with the property that the net $\{\mu'_f : f \in \mathscr{S}\}$ is consistent and such that $\langle \xi'_0, \varphi'(f_1)\varphi'(f_2)\ldots\varphi'(f_{*(f)})\xi'_0\rangle_{\mathscr{H}'} = \langle \xi_0, \varphi(f_1)\varphi(f_2)\ldots\varphi(f_{*(f)})\xi_0\rangle_{\mathscr{H}}$, for all $f \in \mathscr{S}$, then there exists a unitary mapping U of $\mathscr{H}$ onto $\mathscr{H}'$ such that

$$\mathscr{D}' = U\mathscr{D}, \qquad \xi'_0 = U\xi_0$$

and

$$\varphi'(f) = U\varphi(f)U^{-1}, \qquad \text{for all } f \in \mathscr{S}(\mathbb{R}^d).$$

Proof. Since $\{(\mathscr{B}_f, \mu_f): f \in \mathscr{S}\}$ is a net of probability algebras possessing the property that the net $\{\mu_f : f \in \mathscr{S}\}$ of states is consistent relative to the family $\{h_{f^{(2)}f^{(1)}} : (f^{(1)}, f^{(2)}) \in \mathscr{S} \times \mathscr{S}$ with $f^{(1)} \leqslant f^{(2)}\}$ of canonical $*$-homomorphic imbeddings it follows from theorem (3.9) that there exists a probability algebra $(\mathfrak{A}, \mu)$ and a family $\{\phi_f : f \in \mathscr{S}(\mathbb{R}^d)\}$ of $*$-homomorphisms of $\mathfrak{A}$ into $\mathscr{B}$ such that

$$\phi_f(\boldsymbol{a}) \in \mathscr{B}_f \qquad \text{for } \boldsymbol{a} \in \mathfrak{A} \text{ and } f \in \mathscr{S}(\mathbb{R}^d).$$

Let

$$\mathfrak{A}_\mu = \{\boldsymbol{a} \in \mathfrak{A} : \mu(\boldsymbol{a}^*\boldsymbol{a}) = 0\}.$$

Then $\mathfrak{A}_\mu$ is a left ideal of $\mathfrak{A}$.

Set $\mathfrak{A}/\mathfrak{A}_\mu = \mathscr{D}$. Then $\mathscr{D}$ is the complex linear space of all equivalence classes of $\mathfrak{A}$ modulo $\mathfrak{A}_\mu$. If $\boldsymbol{a} \in \mathfrak{A}$, denote the equivalence class of $\boldsymbol{a}$ by $[\boldsymbol{a}]$. Introduce an inner product $\langle\,,\rangle$ on $\mathscr{D}$ by

$$\langle [\boldsymbol{a}], [\boldsymbol{b}]\rangle = \mu(\boldsymbol{a}^*\boldsymbol{b})$$

for arbitrary $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{A}$. This inner product is well defined.

Let $\mathscr{H}$ be the completion of $\mathscr{D}$ in the norm-topology induced by $\langle\,,\rangle$. Then $\mathscr{H}$ is a Hilbert space with inner product $\langle\,,\rangle_{\mathscr{H}}$. It is the Hilbert space $(\mathscr{H}, \langle\,,\rangle_{\mathscr{H}})$ whose existence was asserted in the theorem.

For $f \in \mathscr{S}$, with $f = (f_1, f_2, \ldots, f_{*(f)})$, let $\tilde{g}_f$ be the canonical $*$-homomorphic imbedding of $\mathscr{B}_f$ in $\mathfrak{A}$. Let $\boldsymbol{a}_0$ be a fixed member of $\mathfrak{A}$ and set

$$(\phi_{f_1}(\boldsymbol{a}_0)\phi_{f_2}(\boldsymbol{a}_0)\ldots\phi_{f_{*(f)}}(\boldsymbol{a}_0), \mathbf{1}, \ldots, \mathbf{1}) = b_0(f) \in \mathscr{B}_f.$$

Then $b_0(f)$ is a fixed member of $\mathscr{B}_f$.

For each $f \in \mathscr{S}(\mathbb{R}^d)$, we now define a linear operator $\varphi(f)$ on $\mathscr{D} \subset \mathscr{H}$ through the following prescription:

$$\varphi(f_1)\varphi(f_2)\ldots\varphi(f_{*(f)})[\boldsymbol{c}] = [\tilde{g}_f(b_0(f))\boldsymbol{c}],$$

for arbitrary $(\boldsymbol{c}, f) \in \mathfrak{A} \times \mathscr{S}$. Thus in particular, for $f \in \mathscr{S}(\mathbb{R}^d)$,

$$\varphi(f)[\boldsymbol{c}] = [\tilde{g}_f(b_0(f))\boldsymbol{c}],\ \boldsymbol{c} \in \mathfrak{A}.$$

Notice that in view of equation (4.1)

$$\langle \varphi(f)[c], [d]\rangle_{\mathscr{H}} = \langle [c], \varphi(\bar{f})[d]\rangle_{\mathscr{H}}$$

for all $\boldsymbol{c}, \boldsymbol{d} \in \mathfrak{A}$. Moreover, it is evident that $\mathscr{D}$ is a common dense invariant domain for the family $\{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\}$, together with their adjoints, of linear operators on $(\mathscr{H}, \langle\,,\rangle_{\mathscr{H}})$. It is the family $\{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\}$ whose existence was asserted.

Set $[\mathbb{1}] = \xi_0$, where $\mathbb{1}$ is the identity element of $\mathfrak{A}$. Then $\xi_0 \in \mathscr{D}$, and it is the vector whose existence was asserted.

To conclude the proof of existence, let $f \in \mathscr{S}$, with $f = (f_1, f_2 \ldots, f_{*(f)})$. Then

$$\begin{aligned}\langle \xi_0, \varphi(f_1)\varphi(f_2)\ldots\varphi(f_{*(f)})\xi_0\rangle_{\mathscr{H}} &= \langle [\mathbb{1}], [\tilde{g}_f(b_0(f))]\rangle_{\mathscr{H}} \\ &= \mu(\tilde{g}_f(b_0(f))) = (\mu \circ \tilde{g}_f)(b_0(f)) \\ &= \mu_f(b_0(f)),\end{aligned}$$

since

$$\mu \circ \tilde{g}_f = \mu_f, \qquad f \in \mathscr{S}.$$

Finally, we omit the proof of the uniqueness part because it is standard. □

4.5. Remark. The quadruplet $((\mathscr{H}, \langle\,,\,\rangle_{\mathscr{H}}), \mathscr{D}, \xi_0, \{\varphi(f): f \in \mathscr{S}(\mathbb{R}^d)\})$ described in the above theorem already satisfies some of the Wightman axioms for quantum fields. If we now require that the quadruplet satisfies the remaining Wighman axioms, then we have a scalar quantum field. Thus our theory of non-commutative stochastic processes affords, in particular, another way of discussing quantum fields. In the study of quantum fields, a number of different approaches have been considered (Segal 1963, Haag and Kastler 1964, Nelson 1973, Velo and Wightman 1973). However, in view of the inherently probabilistic nature (in an entirely non-classical sense) of quantum theory, a genuinely stochastic description of quantum fields, such as we present here, seems to us not only desirable but also worthwhile.

Acknowledgment

The author is grateful to the two referees of this paper for their stimulating comments.

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